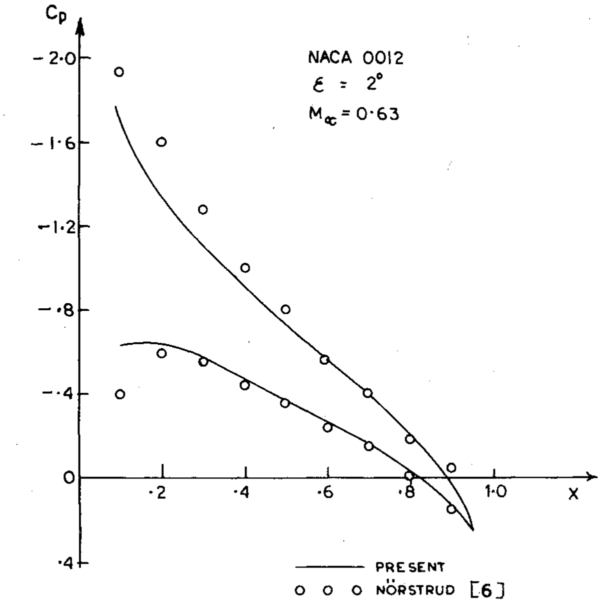

Fig. 3 Reduced pressure coefficient at NACA 0012 profile at incidence.

For evaluating the solution, Eq. (15), it is necessary to compute the integral part of U_A given by Eq. (9), which contains a double integral over the known function U_p^-. To find an approximate value of the double integral we make the substitution

$$U_p^-(X,Y) = U_p^-(X,0)\;[1+Y/b(X)]^{-2} \qquad (20)$$

where $b(x)$ is an unknown parameter to be determined by the irrotationality condition. A similar substitution has been previously used by many authors[2] and is known to be a good one. Substituting Eq. (20) in Eq. (9), carrying out the indicated integration with respect to η, and noting the observations made therein[2] we get

$$U_A = (U_p^-)^2/2 - \int_0^1 \frac{[U_p^-(\xi,0)]^2}{2\,b(\xi)} \cdot E\left(\frac{\xi - x}{b}\right) \mathrm{d}\xi \qquad (21)$$

where

$$E(x) = \frac{4}{\pi}\,\frac{1}{(1+X^2)^5}\Big[\frac{\pi}{2}\,(5-10X^2+X^4)\,|X| - (1-10X^2+5X^4)\ln|X| - \frac{1}{12}\,(25-71X^2-X^4-X^6)\,(1+X^2)\Big] \qquad (22)$$

For performing the integration in Eq. (21), Simpson's one third rule has been used, the singularities at $\xi = X$, 0, and 1 appropriately being taken care of.

The reduced pressure coefficient $C_p = -2U(X,0)$ for a parabolic arc profile at small incidence has been computed by the preceding method and compared with theoretical results[7] and with the experimental results of Knechtel[10] in Figs. 1 and 2, respectively, for subcritical and supercritical flow. From the figures, it is apparent that the present solution agree well with the experimental results. It should be mentioned that the agreement of the present solution with the experimental results of Knechtel for a parabolic arc profile at an incidence of 4° is particularly noteworthy (Fig. 2). The reduced pressure coefficients for NACA 0012 profile for different freestream Mach number M_∞ and angle of incidence ϵ have been presented in Figs. 3 and 4, where Fig. 4 shows a supercritical finite difference solution of the integral equation by Nörstrud.[6] The agreements are excellent. Thus it appears that the present

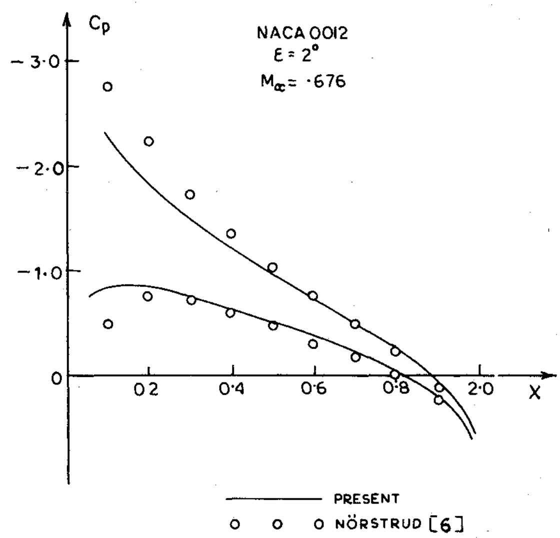

Fig. 4 Reduced pressure coefficient at NACA 0012 profile at nonzero incidence.

solution is capable of delivering satisfactory results at a very low expenditure of labor. To compute one example of a lifting profile by this method takes only 5 minutes on an IBM 1130 electronic digital computer.

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Mean-Square Response of Beams to Nonstationary Random Excitation

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Introduction

VIBRATION of elastic structures to random loads is of considerable interest to the design engineer. The early study of the response of beams and plates to random ex-

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citation has been carried out by Eringen[1] and Samuel and Eringen.[2] But during the past decade, several authors have contributed in the field of random vibration of mechanical systems.[3] Free vibration of stochastic beam has been considered by Hoshiya and Shah.[4] Most of this early work has been confined to stationary processes because of their relative mathematical simplicity. More recently, attention is being paid to determine the response of mechanical systems to nonstationary random excitation. Holman and Hart[5] studied the response of a first-order system to a special type of random loading, namely, a segmented nonstationary random excitation. Barnoski and Maurer[6] considered a second-order system which was also considered by Bucciarelli and Kuo[7] but, in the latter case, the authors derived quite simplified expressions by assuming light damping in the system.

In the present work the response of an elastic beam under nonstationary random loading has been studied. Following arguments similar to that of Bucciarelli and Kuo,[7] the expression for the mean-square response of the beam has been greatly simplified. Some specific examples using a simply supported beam have been provided and the results plotted.

Basic Equation

According to the Euler-Bernoulli theory, the transverse vibration of an elastic beam is governed by the following equation

$$EI(\partial^4 u/\partial x^4)+\rho(\partial^2 u/\partial t^2)+C(\partial u/\partial t)=f(x,t) \qquad (1)$$

where $u(x,t)$ = transverse deflection, E = Young's Modulus, I = moment of inertia of beam cross-section, ρ = mass/unit length, C = damping coefficient/unit length, and $f(x,t)$ = forcing function.

We assume a series solution of the form

$$u(x,t)=\sum_{n=1}^{\infty} A_n(t)\phi_n(x);\quad n=1,2,3\ldots \qquad (2)$$

where $\phi_n(x)$ are the normal modes of the natural vibration of the beam. Substituting Eq. (2) into Eq. (1), and taking inner product with $\phi_m(x)$, noting that

$$\int_0^\ell \phi_n(x)\phi_m(x)=0 \quad m\neq n$$

$$\neq 0 \quad m=n$$

we obtain

$$EI\,\lambda_m^4 A_m+\rho\ddot{A}_m+C\dot{A}_m=P_m(t) \qquad (3)$$

where

$$P_m(t)=\langle\phi_m(x)\,|f(x,t)\rangle/\langle\phi_m(x)\,|\,\phi_m(x)\rangle \qquad (4)$$

and $\langle\phi_m\,|f\rangle$ denotes the inner product; $m=1,2,3,\ldots$.

The nonstationary forcing function is assumed to be separable in the following form

$$f(x,t)=F(x)G(t)g(t) \qquad (5)$$

where $G(t)$ is a deterministic "slowly varying" function of time and $g(t)$ is a stationary random function with zero mean. It should be noted that under these assumptions, $f(x,t)$ becomes a nonstationary random forcing function.

Equation (4) now becomes

$$P_m(t)=\frac{\langle\phi_m(x)\,|F(x)\rangle}{\langle\phi_m(x)\,|\,\phi_m(x)\rangle}\,G(t)g(t) \qquad (6)$$

Equation (3) may be rewritten as

$$\ddot{A}_m+2\xi_m\omega_m\dot{A}_m+\omega_m^2 A_m=\alpha_m G(t)g(t) \qquad (7)$$

where

$$\xi_m=C/2\rho\omega_m \text{ or } \xi_m\omega_m=C/2\rho=B \qquad (8)$$

$$\omega_m^2=EI\,\lambda_m^4/\rho \qquad (9)$$

$$\alpha_m=(1/\rho)\;\frac{\langle\phi_m(x)\,|F(x)\rangle}{\langle\phi_m(x)\,|\,\phi_m(x)\rangle} \qquad (10)$$

The general solution of Eq. (7) can be obtained by means of Duhamel's integral. Assuming the system to be at rest initially, i.e., $A_m=0$ for $t<0$, we obtain

$$A_m(t)=(\alpha_m/\Omega_m)\int_0^t e^{-B(t-t_1)}$$

$$\times\sin\Omega_m(t-t_1)G(t_1)g(t_1)\,dt_1 \qquad (11)$$

where $\Omega_m=\omega_m(1-\xi_m^2)^{1/2}$. Substituting Eq.(11)into Eq.(2) we obtain

$$u(x,t)=\sum_{m=1}^{\infty}(\alpha_m\phi_m(x)/\Omega_m)\int_0^t e^{-B(t-t_1)}$$

$$\times\sin\Omega_m(t-t_1)G(t_1)g(t_1)dt_1 \qquad (12)$$

The mean-square response at any point of the beam, which is the expected value of $u(x,t)\circ u(x,t)$ may be written as

$$E[u^2(x,t)]=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi_m(x)\phi_n(x)$$

$$E[A_m(t_1)A_n(t_2)] \qquad (13)$$

Since $G(t)$ is a well-defined deterministic function, the expected value of $A_m(t)$ is

$$E[A_m(t_1)A_n(t_2)]=\frac{\alpha_m\alpha_n}{\Omega_m\Omega_n}\int_0^t\int_0^t e^{-B(t-t_1)}\sin\Omega_m(t-t_1)$$

$$\times e^{-b(t-t_2)}\sin\Omega_n(t-t_2)G(t_1)G(t_2)E[g(t_1)g(t_2)dt_1dt_2 \qquad (14)$$

where

$$E[g(t_1)g(t_2)]=R(t_1-t_2)=\text{autocorrelation function of } g(t) \qquad (15)$$

Hence the mean-square response at any point along the beam length and at any instant of time t, is given by

$$E[u^2(x,t)]=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[\frac{\alpha_m\alpha_n\phi_m(x)\phi_n(x)}{\Omega_m\Omega_n}$$

$$\times\int_0^t\int_0^t e^{-B(t-t_1)}\sin\Omega_m(t-t_1)e^{-B(t-t_2)}$$

$$\times\sin\Omega_n(t-t_2)G(t_1)G(t_2)R(t_1-t_2)dt_1dt_2\Big] \qquad (16)$$

With the boundary conditions specified and for given autocorrelation function, the mean-square response may be evaluated. However, the task for evaluation could be considerable and, under certain assumptions, Eq. (16) may be greatly simplified to obtain approximate mean-square response of the beam.

Approximate Mean-Square Response

For very light damping, the idea of normal modes may be applied. According to this, the terms contributing most to the mean square response are those for which $m=n$. Thus, Eq. (16) may be simplified to

$$E[u^2(x,t)] \simeq \sum_{m=1}^{\infty}\left[\frac{\alpha_m^2\phi_m^2(x)}{\Omega_m^2}\times\int_0^t\int_0^t e^{-B(t-t_1)}\sin\Omega_m(t-t_1)e^{-B(t-t_2)}\times\sin\Omega_m(t-t_2)G(t_1)G(t_2)R(t_1-t_2)dt_1dt_2\right] \quad (17)$$

It should be noted that the terms of Eq. (17) vary inversely with Ω_m^2 which, again for light damping, indicates the contribution of higher terms to be small compared to that of the first. Finally, Eq. (17) may be simplified to yield

$$E[u^2(x,t)] \simeq \frac{\alpha_1^2\phi_1^2(x)}{\Omega_1^2}\times\int_0^t\int_0^t e^{-B(t-t_1)}\sin\Omega_1(t-t_1)e^{-B(t-t_2)}\times\sin\Omega_1(t-t_2)G(t_1)G(t_2)R(t_1-t_2)dt_1dt_2 \quad (18)$$

Response of a Simply Supported Beam to Shaped White Noise

The autocorrelation function is defined as

$$R(t_1-t_2)=2\pi S_0\delta(t_1-t_2) \quad (19)$$

For a simply supported beam

$$\phi_m(x)=\sin m\pi x/L;\ \lambda_m=m\pi/L;\ \omega_m=(m\pi/L)^2(EI/\rho)^{1/2} \quad (20)$$

If the force $G(t)$ is applied at a point $x=a$ on the beam, then

$$\langle\phi_m(x)|F(x)\rangle=\int_0^L\sin\frac{m\pi x}{L}\,\delta(x-a)dx=\sin\frac{m\pi a}{L}$$

and

$$\langle\phi_m(x)|\phi_m(x)\rangle=\int_0^L\sin\frac{m\pi x}{L}\sin\frac{m\pi x}{L}dx=\frac{L}{2}$$

Hence

$$\alpha_m=\frac{2}{\rho L}\sin\frac{m\pi a}{L} \quad (21)$$

Substituting Eqs. (19-21) into Eq. (17), we obtain

$$E[u^2(x,t)] \simeq \sum_{m=1}^{\infty}\frac{4\sin^2(m\pi a/L)\sin^2(m\pi x/L)}{(\rho L)^2\Omega_m^2}\times 2\pi S_0\times\int_0^t e^{-2B(t-t_1)}\sin^2\Omega_m(t-t_1)G^2(t_1)dt_1 \quad (22)$$

For light damping and $G(t)$ being a slowly varying function, we can use similar arguments as that in Ref. 7 for simplifying Eq. (22) to yield

$$E[u^2(x,t)] = \sum_{m=1}^{\infty}\frac{4\pi S_0\sin^2(m\pi a/L)\sin^2(m\pi x/L)}{(\rho L\Omega_m)^2}\times\int_0^t e^{-2B(t-t_1)}G^2(t_1)dt_1 \quad (23)$$

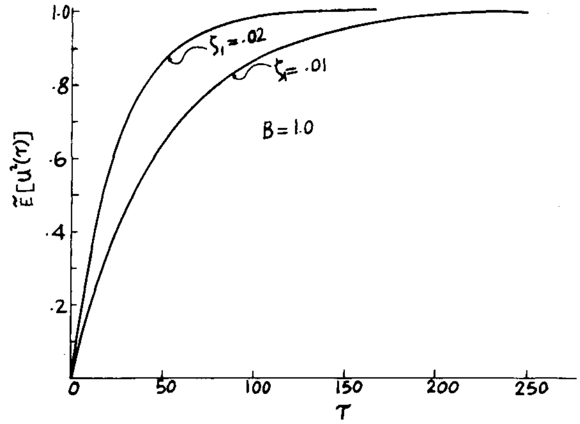

Fig. 1 Mean-square response—unit step × white noise input.

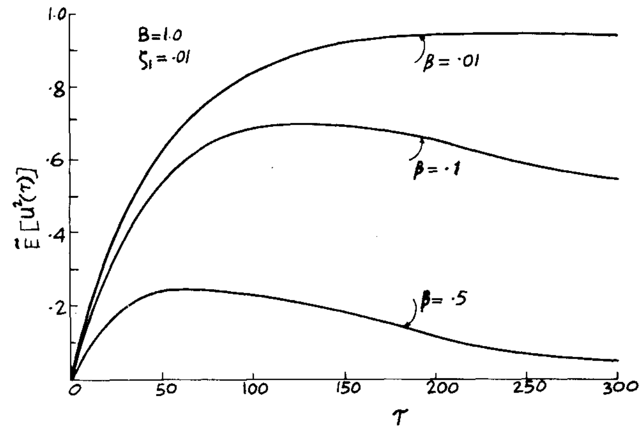

Fig. 2 Mean-square response—exponentially decay function × white noise input.

1) If $G(t)$ is taken as a unit step, the mean-square response from Eq. (23) is

$$E[u^2(x,t)] = \sum_{m=1}^{\infty}\frac{2\pi S_0\sin^2(m\pi a/L)\sin^2(m\pi x/L)}{(\rho L\Omega_m)^2B}(1-e^{-2Bt}) \quad (24)$$

Nondimensional form of Eq. (24) is

$$\tilde{E}[u^2(x,t)]=E[u^2(x,t)]\times 1\Big/\left[\sum_{m=1}^{\infty}\frac{2\pi S_0\sin^2 m\pi a'\sin^2 m\pi x'}{(\rho L\Omega_m)^2B}\right]=(1-e^{-2\xi_1\tau}) \quad (25)$$

where $a'=a/L$, $x'=x/L$, and $\tau=\omega_1 t$. The mean-square response [Eq. (25)] has been plotted in Fig. 1.

2) If $G(t)$ is taken to be an exponentially decaying function, i.e., $G(t)=e^{-\beta t}$ where β is not allowed to be too large so that $G(t)$ remains slowly varying, then

$$E[u^2(x,t)] = \sum_{m=1}^{\infty}\frac{2\pi S_0\sin^2(m\pi a/L)\sin^2(m\pi x/L)}{(\rho L\Omega_m)^2(B-\beta)}(e^{-2Bt}-e^{-2B}) \quad (26)$$

Again the nondimensional form would be

$$\tilde{E}[u^2(\tau)]=E[u^2(x,t)]\times 1\Big/\left[\sum_{m=1}^{\infty}\frac{2\pi S_0\sin^2 m\pi a'\sin^2 m\pi x'}{(\rho L\Omega_m)^2(B-\beta)}\right]=(e^{-(2\beta/\omega_1)\tau}-e^{-2\xi_1\tau}) \quad (27)$$

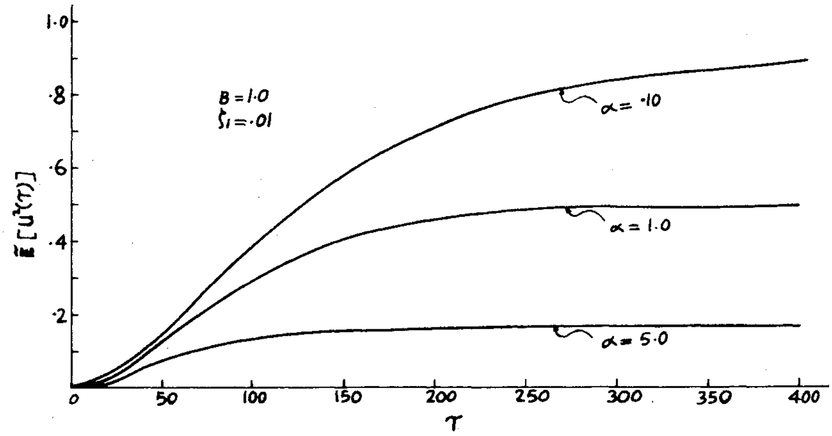

Fig. 3 Mean-square response—unit step × narrow band input.

Equation (27) has been plotted in Fig. 2 for various values of β.

Response of a Simply Supported Beam to Shaped Narrow Band Excitation

We assume that for nth mode the correlation function is of the form

$$R(t_1-t_2)=\sum_{n=1}^{\infty} R_n \exp^{-\alpha|(t_1-t_2)|} \cos W_n(t_1-t_2) \tag{28}$$

where W_m is the center frequency of the exponentially decaying harmonic function.

From Eqs. (17, 20, 21, and 28) we obtain

$$E[u^2(x,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Bigg[\frac{4\sin^2(m\pi a/L)\sin^2(m\pi x/L)}{(\rho L)^2\Omega_m^2} \times\int_0^t\int_0^t \exp^{-B(t-t_1)}\sin\Omega_m(t-t_1)\exp^{-B(t-t_2)} \times\sin\Omega_m(t-t_2)G(t_1)G(t_2)R_n\exp^{-\alpha|(t_1-t_2)|} \times\cos W_n(t_1-t_2)dt_1dt_2\Bigg] \tag{29}$$

We assume that the center frequency of the excitation band coincides with the damped natural frequency of the mth mode, i.e., $\Omega_m=W_m$. Again, following the argument of Bucciarelli and Kuo[7] for light damping, Eq. (29) may be reduced to

$$E[u^2(x,t)]=\sum_{m=1}^{\infty}\Bigg[\frac{4R_m\sin^2(m\pi a/L)\sin^2(m\pi x/L)}{(\rho L)^2\Omega_m^2}e^{-2Bt} \times\int_0^t\int_0^t \exp^{B(t_1+t_2)}\exp^{-\alpha|(t_1-t_2)|}G(t_1)G(t_2)dt_1dt_2\Bigg] \tag{30}$$

For nondimensional form of mean-square response, we may write

$$\tilde{E}[u^2(t)]=E[u^2(x,t)] \times 1\Bigg/\Bigg[\sum_{m=1}^{\infty}\frac{4R_m\sin^2 m\pi a'\sin^2 m\pi x'}{(\rho L\Omega_m)^2}\Bigg] =e^{-Bt}\int_0^t\int_0^t e^{B(t_1+t_2)} \times e^{-\alpha|t_1-t_2|}G(t_1)G(t_2)dt_1dt_2 \tag{31}$$

Unit step: if we take $G(t)$ to be a unit step, Eq. (31) reduces to

$$\tilde{E}[u^2(\tau)]=\Bigg[\frac{1}{B(B+\alpha)}-\frac{2e^{-(\xi_1+\alpha/\omega_1)\tau}}{B^2-\alpha^2}+\frac{e^{-2\xi_1\tau}}{B(B-\alpha)}\Bigg] \tag{32}$$

For the special case $\alpha=B$, we can use L'Hospital's rule on the indeterminate forms of Eq. (32). Thus

$$\tilde{E}[u^2(\tau)]=(1/2B^2)[1-(1+2\xi_1\tau)e^{-2\xi_1\tau}] \tag{33}$$

The mean-square response represented by Eq. (32) has been shown graphically in Fig. 3 for various values of α.

Conclusion

The analysis for the mean-square response becomes simplified considerably under the assumptions of light damping and the envelope function $G(t)$ being slowly varying. The mean-square response reduces to a product of a series summation dependent on x and a time-dependent part, the latter being due to the nonstationary nature of the input force. A numerical check will show that the first term of the series summation is dominant compared to the rest and this simplification would be of premium to design engineers.

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